Appendix B

The Karamata Theorem

In Chapter 3, to obtain the asymptotic distribution of eigenvalues, we applied the following Tauberian theorem due to Karamata. For the proof we refer to [44, Theorem 10.3].
We prove also a weaker version which we have not been able to find in the literature.

Let μ a positive Borel measure on $[0, \infty)$ such that

$$\hat{\mu}(t) = \int_0^\infty e^{-tx} d\mu(x) < \infty$$

for all $t > 0$. The function $\hat{\mu} : (0, \infty) \to \mathbb{R}$ is called the Laplace Transform of μ. The theorem relates the asymptotic behavior of $\mu([0, x])$ as $x \to \infty$ to the asymptotic behavior of $\hat{\mu}(t)$ as $t \to 0$.

Theorem B.0.11. *Let $r \geq 0$, $a \in \mathbb{R}$. The following are equivalent:*

(i) $\lim_{t\to 0} t^r \hat{\mu}(t) = a$;

(ii) $\lim_{x\to\infty} x^{-r} \mu([0, x]) = \dfrac{a}{\Gamma(r+1)}$

where Γ is the Euler's Gamma Function.

We have also used the following weaker version of the previous theorem which we have not been able to find in the literature. In the proposition below we fix a nonnegative, nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\exp\{-\lambda_n t\} \in l^1(\mathbb{R})$ for every $t > 0$.

Proposition B.0.12. *Let $r > 0$, $C_1 > 0$ such that*

$$\limsup_{t\to 0} t^r \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \leq C_1. \tag{B.1}$$

Then

$$\limsup_{\lambda\to\infty} \lambda^{-r} N(\lambda) \leq C_1 \frac{e^r}{r^r}.$$

Moreover if (B.1) holds and

$$\liminf_{t\to 0} t^r \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \geq C_2 \tag{B.2}$$

for some $C_2 > 0$ then

$$\liminf_{\lambda\to\infty} \lambda^{-r} N(\lambda) \geq C_3$$

for some positive C_3.

Proof. Let us suppose that B.1 holds. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $t \leq \delta$

$$\sum_{n\in\mathbb{N}} e^{-\lambda_n t} \leq \frac{C_1 + \varepsilon}{t^r}.$$

This implies that for $\lambda > 0$

$$N(\lambda)e^{-\lambda t} = \sum_{\lambda_n \leq \lambda} e^{-\lambda t} \leq \sum_{n\in\mathbb{N}} e^{-\lambda_n t} \leq \frac{C_1 + \varepsilon}{t^r}.$$

So

$$N(\lambda) \leq (C_1 + \varepsilon)\frac{e^{\lambda t}}{t^r}$$

in $[0, \delta]$. Minimizing on t in such interval it follows

$$N(\lambda) \leq (C_1 + \varepsilon)\lambda^r \frac{e^r}{r^r}$$

for λ large enough.
Suppose now that (B.1) and (B.2) hold. Then, given $\varepsilon > 0$, for t small enough, we have

$$\frac{C_2 - \varepsilon}{t^r} \leq \sum_{n\in\mathbb{N}} e^{-\lambda_n t} = \sum_{\lambda_n \leq \lambda} e^{-\lambda_n t} + \sum_{\lambda \leq \lambda_n \leq 2\lambda} e^{-\lambda_n t} + \ldots \leq \sum_{k=1}^{\infty} e^{-\lambda(k-1)t} N(k\lambda).$$

We have

$$sN(s\lambda) \geq \sum_{k=1}^{s} e^{-\lambda(k-1)t} N(k\lambda)$$

and, using the upper bound obtained in the first part of the proof, for λ large enough,

$$sN(s\lambda) \geq \frac{C_2 - \varepsilon}{t^r} - C\lambda^r \sum_{k=s+1}^{\infty} e^{-\lambda(k-1)t} k^r.$$

Setting $t = \frac{1}{\lambda}$, then t is small when λ is large enough and one obtains

$$sN(s\lambda) \geq (C_2 - \varepsilon)\lambda^r - C\lambda^r \sum_{k=s+1}^{\infty} e^{-(k-1)} k^r.$$

Choosing now s sufficiently large we obtain

$$sN(s\lambda) \geq C_3 \lambda^r$$

and the proof follows. $\square$

Arguing as in the previous proposition, it is possible to prove the following result.

Proposition B.0.13. *Let $C_1 > 0$ such that*

$$\limsup_{t \to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \leq C_1. \tag{B.3}$$

Then

$$\limsup_{\lambda \to \infty} \lambda^{-\frac{N}{2}} (\log \lambda)^{-\frac{N}{\alpha}} N(\lambda) \leq C_2$$

for some positive C_2. Moreover if (B.3) holds and

$$\liminf_{t \to 0} \frac{t^{\frac{N}{2}}}{(-\log t)^{\frac{N}{\alpha}}} \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \geq C_3 \tag{B.4}$$

for some $C_3 > 0$ then

$$\liminf_{\lambda \to \infty} \lambda^{-\frac{N}{2}} (\log \lambda)^{-\frac{N}{\alpha}} N(\lambda) \geq C_4$$

for some positive C_4.